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On the Law of Gravitation in the Binary Systems.[*]

BY FRANK LOXLEY GRIFFIN.

§ 1. *The Problem.*

That the Newtonian law of attraction operates in the solar system, was long ago shown to be very highly probable; the discovery of binary systems naturally raised the question as to whether the same law operates in them. Any reply to such a question is based, of course, upon both the observational data and certain fundamental assumptions expressed or implied. In this paper the assumptions hitherto used in demonstrating the operation of Newton's law are replaced by others; incidentally a new proof with the old hypotheses is obtained.

The observational data have to do with the apparent orbit, which is merely the projection of the actual orbit upon the plane tangent to the "celestial sphere"; but, combined with the assumption that the actual orbit is a plane curve, they permit the conclusion that the satellite describes an ellipse, subject to the action of a central force directed toward the primary.[†]

These data being accepted, the operation of Newton's law can be established by applying a theorem of Darboux and Halphen[‡] and employing the following assumptions: the force

(I) is a function of the distance alone,
(II) does not vanish at the center of force, and
(III) admits no orbits except conic sections, whatever be the initial conditions.

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[*] Read (in part) before the American Mathematical Society December 28, 1906, and (supplemented) September 6, 1907.

[†] F. Tisserand: *Traité de Mécanique Céleste*, Vol. I, p. 35. In the last analysis, this conclusion is to be regarded as a pure assumption. For, in the first place, observational error is considerable in comparison with the small angles measured; so that the observed positions often deviate widely from even the "most probable ellipse". And secondly, — and more important, — even if absolutely correct observations were possible, any number of them would, without the help of assumptions, be powerless to give any information as to the positions of the star at any instants save those of the observations, or to show that the orbit is a conic with the law of areas exactly fulfilled.

[‡] *Comptes rendus de l'Académie des Sciences* (Institut de France), Vol. LXXXIV, pp. 760–62, 936–38, 939–41; cf. also Tisserand, *loc. cit.*, pp. 36–42.

Another demonstration uses the following assumptions with a theorem of Bertrand:* the force

(I) is a function of the distance alone, and
(I a) is a single-valued function,†
(II) does not vanish at the centre of force, and
(III) admits no orbits except closed curves whatever be the initial conditions, provided that the velocity be less than a certain limit.

For the proof given in the present paper the hypotheses are: the force

(I) is a function of the distance alone,
(II) does not vanish at the centre of force, and
(III) varies along the given ellipse according to a law which, if operating throughout the plane, *would permit real motion in some region* outside of any circle about the center of force.

In the present paper, as in the classic discussions, the bodies are regarded as material points. [See § 15, however.]

In comparing these three sets of hypotheses, it is hardly necessary to point out the great severity of the third in each of the classic sets [that of Darboux and Halphen being more severe than that of Bertrand, involving the latter]; or to notice the mildness of the third in the new set, since to deny it is to admit either that the force varies according to different laws in different parts of the plane, or that a body started anywhere outside of a certain circle with real initial conditions would immediately disappear from space. Also (I a), while of the sort readily admitted in considering the forces of nature, is one of the least easily dispensed with in the new set, its absence necessitating an increased use of the remaining hypotheses and a considerably lengthened discussion. It should, however, be noted that (II) can be omitted from either of the classic sets, if to the observational data be added the statement, probably justifiable, that the primary star is not at the center of the ellipse described by the satellite. And further, — because of a theorem due to Bertrand,‡ — Darboux and Halphen, knowing that at least one orbit is an ellipse, do not need that part of the observational data which states that the force is central.

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* *Comptes rendus*, Vol. LXXVII, pp. 849–53; cf. Tisserand, *loc. cit.*, pp. 43–48.

† This is tacitly used by Bertrand in the assumption that the orbit has only one apsidal angle. For a simple example of an orbit with two apsidal angles see a paper by the writer in the *American Mathematical Monthly*, Vol. XIV (1907), pp. 199–201.

‡ *Comptes rendus*, Vol. LXXXIV, pp. 731–32.

The present discussion has three parts: (1) a proof of the operation of the Newtonian law; (2) a treatment of the trajectories admitted by the laws satisfying hypotheses (I) and (III),—an extensive category; and (3) a consideration of various questions relating to sets of hypotheses, including a new proof with the hypotheses of Darboux and Halphen.

The demonstration in part (1) starts from the fact that the force required for the description of the given ellipse as a central orbit varies in a manner dependent upon the position of the center of force in the plane. From the polar equation of the ellipse, the pole being at any point, an expression is obtainable for the required force as a function of the distance alone. An examination of the equation of the ellipse shows that if the center of force be elsewhere than on the major axis, hypothesis (III) is contradicted; and if the center lies elsewhere than at a focus, (II) is not satisfied.

It may be noted at this point that the center of force must lie *within* the given ellipse, since a closed curve can not be described as a central orbit with a center of force exterior to, or upon, the curve.*

§ 2. *Lemma Concerning Imaginary Forces.*

For later reference it is well to observe that if the resultant force—central or not—be imaginary in any portion of space, real motion is impossible in that region. For, suppose a particle (x, y, z) to move there, that is, let

$$x = \phi_1(t), \qquad y = \phi_2(t), \qquad z = \phi_3(t),$$

where the $\phi_i(t)$ are *real* functions. From the definition of a derivative it follows that $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are real functions of t, as are also the second derivatives, or component accelerations. Hence, at every instant, the resultant acceleration and force are real. This contradicts the hypothesis, so that the supposition of real motion must be abandoned.

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*E. J. Routh: *Dynamics of a Particle*, pp. 292–93.

§ 3. *Polar Equation of the Given Ellipse.*

Let the center of force be any point within the given ellipse but not on a principal axis. Let coördinate axes coinciding with the principal axes be so chosen that the center of force (x_1, y_1) lies in the third quadrant. Then the equation of the ellipse is

$$(1-e^2)x^2+y^2=a^2(1-e^2), \tag{1}$$

where a and e are respectively the major semi-axis and the eccentricity. Now from any point (x_1, y_1) of the third quadrant there is a normal to the ellipse (in the first quadrant), where the radius vector is a maximum, and where, if λ denote the angle between the X-axis (positively directed) and the normal (outward directed), $0<\lambda<\frac{\pi}{2}$. Let ρ_1 be the distance from the foot of the normal (x_0, y_0) to (x_1, y_1), and ρ_2 be the normal length to the Y-axis. Then, by hypothesis, $\rho_1>\rho_2$.

Let the center of force be selected now as pole, and the coördinates of any point of the ellipse be u and θ, where u is the reciprocal of the radius vector ρ, and θ is the longitude measured from a line parallel to OX. Then, since

$$x=x_1+\rho\cos\theta, \qquad y=y_1+\rho\sin\theta,$$

equation (1) becomes

$$(1-e^2)(\cos\theta+x_1u)^2+(\sin\theta+y_1u)^2=a^2(1-e^2)u^2. \tag{2}$$

The elimination of $\sin\theta$ gives

$$(\alpha\cos^2\theta+\beta\cos\theta+\gamma)^2=\delta^2(1-\cos^2\theta), \tag{3}$$

where

$$\alpha=-e^2, \quad \beta=2x_1u(1-e^2), \quad \gamma=1+y_1^2u^2+(x_1^2-a^2)(1-e^2)u^2,$$

and

$$\delta=2y_1u.$$

Re-arrangement of (3) yields the desired form:

$$\cos^4\theta+4B\cos^3\theta+6C\cos^2\theta+4D\cos\theta+E=0, \tag{4}$$

where

$$B=\frac{\beta}{2\alpha}, \quad C=\frac{\beta^2+2\alpha\gamma+\delta^2}{6\alpha^2}, \quad D=\frac{\beta\gamma}{2\alpha^2}, \quad E=\frac{\gamma^2-\delta^2}{\alpha^2}.$$

§ 4. θ *as a Function of* u.

It is necessary to examine the solutions of the quartic equation (4).* Now this equation is satisfied by $\cos\theta = \cos\lambda$ if $u = u_1 = \frac{1}{\rho_1}$. Since u is a minimum at $\theta = \lambda$, there are not only values of θ greater than λ but also values of θ less than λ, for which u has any value slightly greater than u_1. Hence equation (4) defines $\cos\theta$ as a multiple-valued function of u; that is, $\cos\theta$ is one function of u one side of the maximum normal and another function the other side.

But $\cos\theta$ is an *algebraic* function of u; viz.:

$$\cos\theta = -B \pm \sqrt{p} \pm \sqrt{q} \pm \sqrt{r}, \tag{5}$$

in which p, q and r are the roots of Euler's cubic, and the product of the radicals has the sign of $-G$, where $G = D - 3BC + 2B^3 = -\frac{\beta\delta^2}{4\alpha^3}$. Since x_1 and y_1 are negative, so are β and δ and consequently G. Hence either all three radicals in (5) must be positive or one positive and two negative; also, none of the roots p, q, r, can vanish. These latter are given by

$$\begin{aligned} p &= B^2 - C + \tfrac{1}{2}\sqrt[3]{-J+\sqrt{M}} + \tfrac{1}{2}\sqrt[3]{-J-\sqrt{M}}, \\ q &= B^2 - C + \tfrac{\omega}{2}\sqrt[3]{-J+\sqrt{M}} + \tfrac{\omega^2}{2}\sqrt[3]{-J-\sqrt{M}}, \\ r &= B^2 - C + \tfrac{\omega^2}{2}\sqrt[3]{-J+\sqrt{M}} + \tfrac{\omega}{2}\sqrt[3]{-J-\sqrt{M}}, \end{aligned} \tag{6}$$

where in each case the radical denotes the principal root, and where

$$\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}, \text{ and } -27M = I^3 - 27J^2 \text{ (the discriminant)},$$

I and J being certain integral functions of the coefficients B, C, D and E.

Now since $\cos\theta$ is an algebraic function of u, it can be double-valued only by some radical having different signs on the two sides of the maximum normal. This requires that one of the outer radicals $\sqrt{p}$, $\sqrt{q}$, or $\sqrt{r}$ shall change sign; for in the definitions (6) each sign has been definitely chosen. But, since G is negative always, this requires that two of the radicals shall change sign. Moreover, in the ellipse $\cos\theta$ is a continuous function of the continuous variable u, and as p, q and r can not vanish, the two radicals which change must be always of opposite signs, and their difference must vanish for $u = u_1$.

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* In this section the notation is that of Burnside and Panton in *Theory of Equations*, p. 121.

Now these two radicals are $\sqrt{q}$ and $\sqrt{r}$, and these are distinct from $\sqrt{p}$ for $u = u_1$, which follows from conditions for equal roots of Euler's cubic: $I^3 - 27 J^2 = 0$ for two equal roots, and $I = J = 0$ for three equal roots.* It has been seen that the first condition must be satisfied for $u = u_1$; (6) then shows that q and r become equal. That the latter conditions are not satisfied may be seen by a laborious direct computation of I or J, which may be avoided as follows: If $p = q = r$ for $u = u_1$, the quartic equation (4) will have a triple root, $\cos\theta = \cos\lambda$, as three of the four possible combinations of signs in (5) yield the same value. But while $\cos\theta = \cos\lambda$ is a double solution regardless of the position of the origin on the normal, the solution is triple only if the origin lie on the major axis or at the center of curvature for the given normal.† Both of these possibilities are excluded, since $\rho_1 > \rho_2 >$ radius of curvature $>$ normal length to x-axis. Hence, for the positions of the center of force at present admitted, $I \cdot J \neq 0$ for $u = u_1$.

Finally, the proper signs being chosen, (5) becomes

$$\cos\theta = -B - \sqrt{p} \pm (\sqrt{q} - \sqrt{r}), \tag{7}$$

where one sign is required when $0 < \theta < \lambda$, and the other sign when $\lambda < \theta < \frac{\pi}{2}$.

The next step of importance is to show that the difference $\sqrt{q} - \sqrt{r}$ contains the factor $(u - u_1)^{\frac{1}{2}}$; *i. e.*, $\sqrt{q} - \sqrt{r} = \sqrt{u - u_1} \cdot \psi(u)$ where $\underset{u=u_1}{L}\, \psi(u) \neq 0, \infty$. For this purpose let $\sqrt{q} - \sqrt{r}$ be expressed as a fraction, thus:

$$\sqrt{q} - \sqrt{r} = \frac{\sqrt{-3M}}{(\sqrt{q} + \sqrt{r})\left[(-J + \sqrt{M})^{\frac{2}{3}} + \frac{I}{3} + (-J - \sqrt{M})^{\frac{2}{3}}\right]},$$

which shows, since every term in each factor of the denominator is positive at $u = u_1$, that $\sqrt{q} - \sqrt{r}$ contains the factor $u - u_1$ to the same power as does $\sqrt{M}$,

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* Burnside and Panton, *loc. cit.*, pp. 124, 144.

† For the circle $u = u_1$ meets the ellipse in only two coincident points at (u_1, λ), unless the circle be the osculating circle for that normal, in which case they meet in three coincident points. If the pole be upon OX, the double solutions $\cos\theta = \cos\lambda$, and $\cos\theta = \cos(-\lambda)$, make either a quadruple solution.

Verification may be obtained by the test with the derived equations after expressing x_1 and y_1 as functions of λ. (See a problem, by the author, in the *American Mathematical Monthly*, Jan. 1909.)

Since M is a polynomial (of degree 12) in u, vanishing for $u = u_1$, some integral power of $u - u_1$ is a factor. To test directly the order of this factor is unnecessary, as it will appear in the next equation.

Differentiate (7) with respect to u, denoting derivatives by primes; substitute $\sqrt[3]{-J+\sqrt{M}} = R_1$, $\sqrt[3]{-J-\sqrt{M}} = R_2$, $\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{r}} = 2U$, and $\frac{1}{\sqrt{q}} + \frac{1}{\sqrt{r}} = \frac{2V}{\sqrt{3}}$, so that $\frac{\omega^2}{\sqrt{r}} - \frac{\omega}{\sqrt{q}} = U - iV$, and $\frac{\omega}{\sqrt{r}} - \frac{\omega^2}{\sqrt{q}} = U + iV$; and re-arrange:

$$\begin{aligned} \sin\theta\frac{d\theta}{du} = B' + \frac{2BB'-C'}{2}\left[\frac{1}{\sqrt{p}} \mp 2U\right] - \frac{J'}{12}\left\{\left(\frac{1}{\sqrt{p}} \pm U\right)\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right. \\ \left. \mp iV\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)\right\} + \frac{M'}{24\sqrt{M}}\left\{\left(\frac{1}{\sqrt{p}} \pm U\right)\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right) \mp iV\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right\}. \end{aligned} \tag{8}$$

Now since the line $\theta = \lambda$ is a normal, $\frac{du}{d\theta} = 0$ at (u_1, λ). The only denominator in (8) which vanishes at $u = u_1$ is $\sqrt{M}$. If M contains the factor $(u-u_1)^\nu$, M' contains the factor $(u-u_1)^{\nu-1}$; hence, to have $\frac{du}{d\theta} = 0$ for $u = u_1$, it is necessary that $\frac{\nu}{2} > \nu - 1$, or $\nu < 2$. Therefore $\nu = 1$. Also, putting $\sqrt{u-u_1} = x$, inspection shows that x occurs to the power unity in U and in $\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)$, and to the power zero in V, B', $2BB'-C'$, and $\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)$; $(2BB'-C')$ may however vanish identically). Whether J' carries a factor x is immaterial; suppose x appears to the power $(n-2)$.

Then equation (8) may be written in the form

$$\sin\theta\,\frac{d\theta}{du} = \frac{1}{x}\left(g + hx + kx^2 + lx^n\right), \tag{9}$$

where g, h, k and l are functions of u, which do not vanish at $u = u_1$ unless identically zero. Moreover, from (7) one obtains

$$\begin{aligned} \sin^2\theta &= 1 - \left[(B+\sqrt{p})^2 \mp 2(B+\sqrt{p})(\sqrt{q}-\sqrt{r}) + (\sqrt{q}-\sqrt{r})^2\right] \\ &= b + cx - x^2, \end{aligned} \tag{10}$$

where b and c are functions of u, not vanishing with $u - u_1$, and not identically zero since $\theta = \lambda \neq 0, \frac{\pi}{2}$, when $u = u_1$. Hence

$$\left(\frac{du}{d\theta}\right)^2 = x^2 \frac{b + cx - x^2}{(g + hx + kx^2 + lx^n)^2}. \tag{11}$$

The numerator of (11) contains an odd power of x. Even if the numerator is a factor of the denominator, the removal of this factor will still leave the same factor in the denominator, as the latter is a perfect square. And the conjugate factor needed for removing odd powers, if present at all, is squared. Therefore, (11) contains essentially an odd power of x.

Finally, then, for $u < u_1$, x being a pure imaginary, $\left(\frac{du}{d\theta}\right)^2$ is a complex function of u; thus

$$\left(\frac{du}{d\theta}\right)^2 = \Phi(u) = F_1(u) + i F_2(u), \tag{12}$$

where F_1 and F_2 are real for $u < u_1$, but $i F_2(u)$ is real for $u > u_1$.

§ 5. *Exclusion of Laws for Non-Axial Centers.*

The foregoing results permit a great restriction upon the location of the center of force within the ellipse. As is well known the force necessary for the description of any curve as a central orbit, the center of force being at the pole, is given at all points of the curve by

$$f = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2}\right),$$

where h is the constant of areas. There are an infinite number of laws according to any of which the force may vary to produce a given orbit with a given constant of areas and a given position of the center of force;* but by imposing hypothesis (I) at the start, a single law is obtained for the variation of the force along

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*Cf. F. L. Griffin: Certain Trajectories Common to Different Laws of Central Force, *The Astronomical Journal*, Vol. XXVI (1908), pp. 3–4. Inattention to this fact seems responsible for the use of hypotheses inadequate for the conclusion stated concerning binary systems by Professor Moulton [*Introduction to Celestial Mechanics*, p. 80]. In this connection the writer desires to acknowledge his indebtedness to Professor Moulton for much helpful criticism of this paper, which had its origin in the detection of the error mentioned.

the orbit as a function of u alone. Thus, for the positions of the center of force considered above, the differentiation of (12) gives

$$f = \frac{1}{2} h^2 u^2 [2u + \Phi'(u)].$$

If the force varies according to this law throughout the plane, the resultant force is imaginary when $u < u_1$, that is, everywhere outside of the circle $u = u_1$ [or $\rho = \rho_1$]. By the lemma above, real motion is impossible without that circle, so that hypothesis (III) is contradicted. This law and the corresponding positions of the center are, therefore, excluded; that is, the center of force must lie upon a principal axis.

[It may be observed in passing that the foregoing demonstration applies to exterior points and to those upon the ellipse as well as to interior points, and might have been used to exclude such points, if $0 < \lambda < \frac{\pi}{2}$. Further it excludes points of the minor axis within the evolute (except the center), since from such a point there is a non-axial maximum normal. But in some ellipses the evolute does not contain the entire minor axis.]

§ 6. *Exclusion of Laws for Centers on the Minor Axis.*

Next suppose the center of force to be any point $(0, d)$ of the minor axis Then, choosing that point as the pole and introducing polar coordinates, equation (2) becomes, since $x_1 = 0$, $y_1 = d$:

$$e^2 \rho^2 \sin^2 \theta + 2 d \rho \sin \theta + d^2 + (1 - e^2)(\rho^2 - a^2) = 0. \qquad (13)$$

The solution of (13) involves $\sqrt{(1 - e^2)(d^2 + a^2 e^2 - e^2 \rho^2)}$, and so does the expression for the force (unless $d = 0$), as may be seen by forming $\frac{d^2 u}{d \theta^2}$. Now this radical is imaginary for $\rho > \frac{1}{e} \sqrt{d^2 + a^2 e^2}$, so that real motion is impossible outside a certain circle. Again there is a contradiction of hypothesis (III), restricting the center of force to lie upon the major axis. [It may be noted that points in the line of the minor axis exterior to, or upon, the ellipse are likewise excluded.]

§ 7. *Laws Admitted by Hypotheses (I) and (III).*

Finally let the center of force be any point $(d, 0)$ of the major axis. [There is no loss of generality in restricting d to positive values, as this amounts merely to a choice of the positive direction of an axis.] Equation (2) becomes in this case

$$e^2\rho^2\cos^2\theta - 2d\rho(1-e^2)\cos\theta + (1-e^2)(a^2-d^2) - \rho^2 = 0. \tag{14}$$

The solution of (14) is

$$\cos\theta = \frac{1}{\rho e^2}\left[d(1-e^2) \pm \sqrt{e^2\rho^2 + (d^2 - a^2e^2)(1-e^2)}\right]. \tag{15}$$

If $d > ae$, the radical does not vanish for real values of ρ. Its proper sign is ascertained by substituting the coördinates of a particular point, $\theta = 0$, $\rho = a - d$:

$$1 = \frac{d(1-e^2) \pm (d - ae^2)}{e^2(a-d)} \tag{16}$$

which requires that the lower sign be chosen. Equation (15) may then be written

$$\gamma\cos\theta = \alpha u - \sqrt{\beta u^2 + \gamma}, \tag{17}$$

where

$$\alpha = d(1-e^2), \qquad \beta = (d^2 - a^2e^2)(1-e^2), \qquad \gamma = e^2. \tag{18}$$

If $d < ae$, the radical vanishes; but the corresponding value of ρ, viz., $\rho = \sqrt{-\frac{\beta}{\gamma}}$, is not taken in the ellipse unless $d \overset{=}{<} ae^2$. Tests similar to (16) show that

when $ae^2 \overset{=}{<} d < ae$, the sign of the radical is always $-$, and
when $d < ae^2$, the sign is $+$ near $\theta = 0$, and $-$ near $\theta = \pi$.

[When $d < ae^2$ there are two normals from the pole besides the axis; the sign changes at the foot of each.]

Differentiate (17) twice with respect to θ, putting $R = \sqrt{\beta u^2 + \gamma}$ (positive):

$$\left.\begin{aligned} -\gamma\sin\theta &= \left(\alpha - \frac{\beta u}{R}\right)\frac{du}{d\theta} \\ -\gamma\cos\theta &= \left(\alpha - \frac{\beta u}{R}\right)\frac{d^2u}{d\theta^2} - \frac{\beta\gamma}{R^3}\left(\frac{du}{d\theta}\right)^2 = -\alpha u + R. \end{aligned}\right\} \tag{19}$$

Substituting for $\left(\frac{du}{d\theta}\right)^2$ the value obtained by squaring and adding (17) and (19), one has

$$\left(\alpha - \frac{\beta u}{R}\right)\left(\frac{d^2 u}{d\theta^2} + u\right) - \frac{\beta\gamma}{R}\cdot\frac{\gamma^2 - (\alpha u - R)^2}{(\alpha R - \beta u)^2} = R - \frac{\beta u^2}{R} = \frac{\gamma}{R} \qquad (20)$$

whence

$$u + \frac{d^2 u}{d\theta^2} = \frac{\gamma^2(\beta\gamma + \alpha^2 - \beta)}{(\alpha R - \beta u)^2}. \qquad (21)$$

Multiplication by $h^2 u^2$ gives the required law of force [if $d > ae^2$]

$$f = \frac{\varepsilon u^2}{(\alpha R - \beta u)^3} = \frac{\varepsilon \rho}{(\alpha\sqrt{\beta + \gamma\rho^2} - \beta)^3}, \qquad (22)$$

where

$$\varepsilon = h^2\gamma^3\left(\beta + \frac{\alpha^2 - \beta}{\gamma}\right) = h^2 a^2 e^6 (1 - e^2)^2. \qquad (23)$$

[When $d < ae^2$, the corresponding law in any part of the ellipse is obtained from (22) by assigning to the radical a sign opposite to that specified above for the equation (17).]

The laws represented by (22) have quite different properties according as $d \gtrless ae$; though in both cases f vanishes "at infinity".

Type I, where $d > ae$. Here, since $\beta > 0$, f is everywhere real; also f is everywhere finite since the vanishing of the denominator would require $\rho^2 = \rho_0^2 \equiv \frac{\beta(\beta - \alpha^2)}{\alpha^2\gamma}$, a negative quantity; and f is everywhere positive [attractive], for $\beta > \alpha\sqrt{\beta + \gamma\rho^2}$ would require $\rho^2 < \rho_0^2$.

Type II, where $d < ae$. Here, since $\beta < 0$, f is imaginary within the circle $\rho^2 = -\frac{\beta}{\gamma}$; also f becomes infinite at a finite distance ρ_0 [only for the law required near $\theta = 0$ when $d < ae^2$, — for the others as in Type I]; and in the same case is negative [repulsive] for $\rho > \rho_0$. An exception is the very important case where $\alpha = 0$: the center of force lies at the center of the ellipse and the law is that of the direct distance.

Another special case of interest is that where $d = ae^2$, the center of force being the center of curvature for the end of the major axis. The statement has been made* that a particle can never, in a finite time, arrive at an apse, the center of whose osculating circle is at the center of force, but must approach the osculating circle asymptotically. The present example is a case to which the statement does not apply, since the time from any point (u_0, θ_0) to the apse $(u_1, 0)$ is given by

$$T = \int_{u_0}^{u_1} \frac{\phi(u)\, d u}{(u - u_1)^{\frac{3}{4}}}$$

where $\phi(u_1) \neq 0, \infty$. This integral is finite by a standard test.

§ 8. *Proof of the Operation of the Newton's Law.*

A property common to the laws of both types, except in special cases, is that the force vanishes at the center of force. For, from (22), it is evident that $f = 0$ when $\rho = 0$, unless $\sqrt{\beta}\,(\pm\, \alpha - \sqrt{\beta}) = 0$ which would require either $d = ae$ or $d = a$. The latter exception has been excluded, since in that case the center of force would lie upon the ellipse. The former exception occurs when the center of force is at a focus; and in this case, since $\beta = 0$, (22) reduces to Newton's law, as it should.

Therefore, as the vanishing of f for $\rho = 0$ contradicts hypothesis (II), the Newtonian law must operate in the given binary system.

[It is evident that the method of this section may be applied equally well to exterior points in the line of the major axis. And, again, the laws of type II, which prescribe an imaginary force in a part of the plane, might have been excluded otherwise by strengthening hypothesis (III) to require the possibility of real motion throughout the plane. Reasons for the choice of the method here adopted will be given in § 15.]

PART II. TRAJECTORIES FOR THE LAWS OF TYPE I.

§ 9. *Stability.*

Any law of types I and II admits the given ellipse as a central orbit, in which, at any instant, there are certain initial conditions. The question naturally arises as to the sort of curves that would be described if the initial conditions were different.

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* Cf. Routh, *loc. cit.*, p. 283. This author probably intended, however, to have his statement understood as applying to only the special types of laws discussed in earlier pages, and not to be taken in the general form in which enunciated.

With the exception of the law of the direct distance, whose trajectories are well known, no law of type II permits real motion in all parts of the plane. These laws will not be discussed further. All laws of type I (including that of Newton, whose simple trajectories will not be discussed here), produce for any real initial conditions real orbits, some of whose properties will now be established.

Consider the law (22) for any chosen value of d, $(ae < d < a)$. If a particle be projected from any point in any direction, with a velocity less than a certain function of the initial distance (the "velocity from infinity"), the orbit is stable. That is, the motion is periodic, there being both a pericenter and an apocenter; and a slight change in the conditions of motion at any instant will produce only a slight change in the apsidal distances.

At $t = t_0$ let $u = u_0$, $R = R_0$; and let the velocity be v_0 and the constant of areas be h_2. Then, if $h_2 \neq 0$, the differential equation of the path is, by (22),

$$h_2^2\left(u + \frac{d^2 u}{d\theta^2}\right) = \frac{\varepsilon}{(\alpha R - \beta u)^3}. \tag{24}$$

Integration of (24) after multiplication by $2\,\frac{du}{d\theta}$ gives

$$h_2^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = k_2 + F(u) \equiv k_2 + \frac{\varepsilon u(2\alpha R - \beta u)}{\alpha^2 \gamma (\alpha R - \beta u)^2}, \tag{25}$$

where k_2 is a constant, and where the left member equals the square of the velocity. It is easily seen that the limit of $k_2 + F(u)$ as u becomes infinite is finite. Hence there is some value u_1, such that, for $u > u_1$, $h_2^2\left(\frac{du}{d\theta}\right)^2 < 0$. But $h_2^2\left(\frac{du}{d\theta}\right)^2 = \left(\frac{d\rho}{dt}\right)^2$. Therefore there is a circle $u = u_1$, within which the radial velocity, — and likewise the motion, — is imaginary for the given initial conditions. This establishes the existence of the pericenter.

From the initial conditions follows the equation

$$k_2 = v_0^2 - F(u_0) = v_0^2 - V_0^2, \tag{26}$$

where V_0 is the velocity acquired by a particle in falling from rest at an infinite distance to the initial point, and is real. The latter statement follows from the fact that ε, α, γ, $(\alpha R_0 - \beta u_0)$, — and hence also $2\alpha R_0 - \beta u_0$, — are positive;

so that $F(u_0) > 0$. Now it is possible to choose $v_0 < V_0$; and for any such choice $k_2 < 0$, and $v^2 < 0$ at $u = 0$ as follows from (25). Therefore $h_2^2 \left(\frac{du}{d\theta}\right)^2 < 0$ for all values of u less than some value u_2; so that the motion is imaginary outside the circle $u = u_2$. Thus the orbit has an apocenter.

Since in all central orbits included between two apsidal distances, the apsidal angle and time are constant, if the force be single-valued as required by (22), the radius vector is a periodic function of the longitude and of the time.

If $h_2 = 0$, then $\frac{d\theta}{dt} = 0$ and $\frac{d^2 r}{dt^2} = \frac{\varepsilon u^2}{(\alpha R - \beta u)^3}$. These two equations show the motion to be a periodic oscillation in the line $\theta = \theta_0$.

From (26) it follows that a small change in v_0 would affect k_2 but slightly. And, since $h_2 = v_0 \rho_0 \sin \psi_0$ (where ψ_0 is the angle between the radius vector and tangent at $t = t_0$), a small change in either v_0 or ψ_0 would produce only a small change in h_2. Hence u_1 and u_2, the values of u for which $\frac{du}{d\theta} = 0$, would change but little. The orbit is stable.

§ 10. *Other Conics as Orbits.*

Another question presents itself: is the given ellipse a solitary conic orbit under this law of force, or are there others, — in particular, other ellipses? It will be shown that there is a double infinitude of elliptic orbits, and that through every point of the plane there passes in every direction one, and only one, conic section which is a possible orbit.

In the first place if any other orbit be an ellipse, the center of force must lie upon the major axis and not between the foci; for, if it were elsewhere, the law would be other than of type I.

Hereafter let the constants of the original ellipse be denoted by the subscript 1, and the corresponding constants for any other elliptic orbit, by the same letters without subscripts. Then the equation of any such orbit is

$$\gamma \cos(\theta - \nu) = \alpha u - \sqrt{\beta u^2 + \gamma}, \qquad (27)$$

ν being an arbitrary constant, the longitude of the pericenter. Also the law under consideration becomes in the new notation:

$$f = \frac{\varepsilon_1 \rho}{(\alpha_1 \sqrt{\beta_1 + \gamma_1 \rho^2} - \beta_1)^3}; \qquad (22_1)$$

and similarly the force acting in the new ellipse is given by

$$f=\frac{\varepsilon\rho}{\alpha\sqrt{\beta+\gamma\rho^2}-\beta)^3}, \tag{28}$$

where $\varepsilon=h^2a^2e^2(1-e^2)^2$, h being the new constant of areas. Hence for all values of ρ, the values of f in (22_1) and (28) are identical; from which it follows that the irrational terms in the two must be identical, and, likewise, the rational terms. The application of this principle to $\sqrt[3]{\frac{\rho}{f}}$ in (22_1) and (28) gives

$$\frac{\alpha\sqrt{\beta}}{\varepsilon^{1/3}}=\frac{\alpha_1\sqrt{\beta_1}}{\varepsilon_1{}^{1/3}},\quad \frac{\alpha\sqrt{\gamma}}{\varepsilon^{1/3}}=\frac{\alpha_1\sqrt{\gamma_1}}{\varepsilon_1{}^{1/3}},\quad \frac{\beta}{\varepsilon^{1/3}}=\frac{\beta_1}{\varepsilon_1{}^{1/3}}. \tag{29}$$

From (29) it follows immediately that

$$\frac{\alpha}{\sqrt{\gamma}}=\frac{\alpha_1}{\sqrt{\gamma_1}},\quad \frac{\beta}{\gamma}=\frac{\beta_1}{\gamma_1},\quad \frac{\varepsilon}{\gamma^3}=\frac{\varepsilon_1}{\gamma_1^3}. \tag{30}$$

That is, in all elliptic orbits which may be possible, the three quantities $\frac{\alpha}{\sqrt{\gamma}}$, $\frac{\beta}{\gamma}$ and $\frac{\varepsilon}{\gamma^3}$ must remain invariant. Let these be denoted by A, B and E, respectively; then $\alpha=Ae$, $\beta=Be^2$ and $\varepsilon=Ee^6$. The substitution of these values in (22_1) gives for the law of force:

$$f=\frac{E\rho}{(A\sqrt{B+\rho^2}-B)^3}, \tag{31}$$

and in (27) gives

$$e\cos(\theta-\nu)=Au-\sqrt{Bu^2+1}; \tag{32}$$

so that any possible elliptic orbit under the law (31) is represented by (32) for some value of e, — the original ellipse for $e=e_1$.

Conversely, it will be shown that for all values of e and ν the curves (32) are conics, — a double infinitude possessing certain properties because of the invariants A and B. And further there are real initial conditions, — determined by two invariants, E and another (obtained later), — for which each member of the family of conics (32) is a possible orbit under the law (31).

§ 11. *The Family of Conics.*

Consider the equation (32) for arbitrary values of e and ν. All the distinct curves which are represented by it are obtained for $0\leqq e<\infty$ and $0\leqq\nu\leqq 2\pi$, the curves obtained for negative values of e being included in these. For, by changing the sign of e and adding $\pm \pi$ to v, the equation is unchanged.

To vary v while keeping e constant does not change the shape of a curve, but merely rotates it in the plane. It is, therefore, sufficient to consider the curves for which $v = 0$. This particular infinitude will hereafter be called *the restricted family*; its geometrical properties are possessed by all families in which v has a fixed value.

If rectangular coördinates be introduced by $\cos\theta = xu$, $\sin\theta = yu$, the equation (32) with $v = 0$ becomes:

$$x^2(1 - e^2) + 2Aex + y^2 = D, \tag{33}$$

where

$$D = A^2 - B = (a_1^2 - d_1^2)(1 - e_1^2) > 0.$$

Equation (33) represents for any value of e a conic section, whose transverse axis is in the x-axis, and whose eccentricity is e. For $e \neq 1$, the center is $\left(-\frac{Ae}{1-e^2},\ 0\right)$, and the square of the major semi-axis is $a^2 = \frac{Be^2 + D}{(1-e^2)^2}$. The apocentral distance in any ellipse is, then,

$$\frac{(Be^2 + D)^{\frac{1}{2}} + Ae}{1 - e^2} \text{ or } \frac{D}{(Be^2 + D)^{\frac{1}{2}} - Ae};$$

similarly the pericentral distance in any conic is $\frac{D}{(Be^2 + D)^{\frac{1}{2}} + Ae}$. The former steadily increases with e, having its smallest value $\sqrt{D}$ for $e = 0$, and becomes infinite as $e \doteq 1$. The latter is a decreasing function of e, having its largest value $\sqrt{D}$ for $e = 0$, and approaches zero as $e \doteq \infty$. Hence every conic of the general family has its pericenter within, and its apocenter without, the circle of the family, $\rho^2 = D$.

Various geometrical properties, belonging chiefly to the restricted family, are readily seen. As neither of the invariants A and B is so well suited to interpretation as two of their combinations, viz., D and $\frac{A}{\sqrt{D}}$, these latter two will be used.

I. THE FIRST INVARIANT, D. *All conics of the family with collinear axes pass through two fixed points;* viz., the points where the circle $\rho^2 = D$ meets the perpendicular to the axial line through the origin. For equation (32) with ν fixed is satisfied by $\rho = \sqrt{D}$, $\theta = \nu \pm \frac{\pi}{2}$, whatever be the value of e. [Another proof rests upon the fact that all members of the restricted family pass through $(0, \pm\sqrt{D})$.]

It is possible to define new families of conics related to the family (32) in which there is a simple interpretation of this and the following invariants. If a new conic be described, having the same eccentricity as some given conic of the restricted family, and having as its transverse axis the segment between the center of the given conic and the pole with respect to the given conic of the line joining the fixed points $(0, \pm\sqrt{D})$, then *this new conic passes through the same two fixed points, whatever the value of e.*[*] To establish the property it is sufficient to note the coördinates of the vertices of the new conic,

$$\left(-\frac{Ae}{1-e^2},\ 0\right) \text{ and } \left(\frac{D}{Ae},\ 0\right).$$

Evidently the auxiliary circle of the new conic cuts the Y-axis where $Y^2 = \frac{D}{1-e^2}$, Y being the mean proportional between the two segments of the axis. Let y be the ordinate in the new conic at $x = 0$. Now, in any ellipse, $y^2/Y^2 = b^2/a^2$; and, in any hyperpola, $y^2/Y^2 = -b^2/a^2$. In either case, $y^2 = Y^2(1-e^2) = D$.

II. THE SECOND INVARIANT, $\frac{A}{\sqrt{D}}$. *In all conics of the restricted family, the slope at either fixed point bears to the slope at the end of the latus rectum a ratio which is constant.* For the latter slope is $\pm e$, while the former is seen from (33) to be $\pm\frac{Ae}{\sqrt{D}}$. The ratio is $\pm\frac{A}{\sqrt{D}}$, which is constant.

If a new conic be described, having the same center and eccentricity as some given conic of the restricted family, and having a focus at the origin, the new conic passes through two fixed points whatever the value of e, so that *its latus rectum bears a constant ratio to the segment between the fixed points of the given family.* For, the major semi-axis of the new conic being $\frac{d}{e}$, its latus rectum is

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[*] The members of one family do not belong to the other (except for a single value of e). For in each family there is but one conic with a given eccentricity, and corresponding conics have axes of different lengths.

$\frac{2d(1-e^2)}{e}$ or $2A$; and the distance between the fixed points $(0, \pm\sqrt{D})$ being $2\sqrt{D}$, the ratio is $\frac{A}{\sqrt{D}}$. Again, since the constant latus rectum lies always in the Y-axis, all the new conics pass through $(0, \pm A)$.

III. AN ALTERNATIVE INVARIANT, B. *The family defined in II intersect the tangents at the vertices of corresponding members of the given family on two straight lines,* $y^2 = B$; *and conversely, on* $y^2 = -B$. These statements may be proved analytically by using the equations of the tangents, or geometrically by observing that the segments into which the major axis of the new conic is divided by the vertex of the old are $\frac{d}{e} - a$ and $\frac{d}{e} + a$. It follows that the tangent is cut by the auxiliary circle of the new conic where $y^2 = \frac{d^2 - a^2e^2}{e^2}$, and hence by the conic where $y^2 = \frac{(d^2 - a^2e^2)(1-e^2)}{e^2} = B$. Similarly for the converse.

IV. *Any two conics of the general family intersect, so that no one lies wholly within any other.* For let the values of ν and e in the two curves be respectively ν_1, e_1 and ν_2, e_2; and let the radii vectores be denoted by ρ_1 and ρ_2. Suppose $e_2 > e_1$. Then, by the remark above concerning the pericentral distance, $a_2 - d_2 < a_1 - d_1$; and hence, when $\theta = \nu_2$, $\rho_2 < \rho_1$, for $a_1 - d_1 \overset{=}{<} \rho_1$.

Case I. When $e_2 < 1$, both orbits have apocenters, and $a_2 + d_2 > a_1 + d_1$. Hence, for $\theta = \nu_2 + \pi$, $\rho_2 > \rho_1$; and, for some value of θ between ν_2 and $\nu_2 + \pi$, the two ellipses must intersect.

Case II. When $e_2 \overset{=}{>} 1 > e_1$, the ellipse has an apocenter, while the other conic extends to infinity. Hence there are values of θ where $\rho_2 > \rho_1$, so that the curves cut.

Case III. When $e_1 \overset{=}{>} 1$, there exist values of θ where $\rho_2 > \rho_1$. For the value of $\theta - \nu$ where any conic of (32) goes to infinity is $\phi = \text{arc cos}\left(-\frac{1}{e}\right)$, showing that $\phi > \frac{\pi}{2}$ and ϕ is a decreasing function of e. Now ρ is finite from $\theta = \nu - \phi$ to $\theta = \nu + \phi$; and since $\phi_1 > \phi_2 > \frac{\pi}{2}$, the range of values of θ for which ρ_2 is finite is smaller than that for which ρ_1 is finite, but is greater than π. Hence at least one of the radii vectores where ρ_2 becomes infinite lies in the region where ρ_1 is finite. Near this, $\rho_2 > \rho_1$.

V. *Through every point there passes in every direction one, and only one, conic of the family* (32). For, let the point be $(u_0,\ \theta_0)$, and let the given direction make with the radius vector to that point the angle ψ_0. Then, in the required curve, $\tan\psi_0 = -u_0\left(\frac{d\theta}{du}\right)_0$, where from (32):

$$-\frac{d\theta}{du} = \frac{A\sqrt{Bu^2+1}-Bu}{\sqrt{Bu^2+1}\,[e^2-(Au-\sqrt{Bu^2+1})^2]^{1/2}}. \tag{34}$$

Hence if any member of the family is to have $\psi = \psi_0$ at $u = u_0$, its eccentricity must have the positive (or zero) value given by

$$e_2^2 = (Au_0-\sqrt{Bu_0^2+1})^2 + \left(A - \frac{Bu_0}{\sqrt{Bu_0^2+1}}\right)^2 u_0^2 \cot^2\psi_0. \tag{35}$$

Evidently $e_2^2 \geqq 0$, so that there is a real positive (or zero) solution. The substitution of $e = e_2$, $\theta = \theta_0$ and $u = u_0$ in (32) determines ν so that the conic will pass through the given point. The equations determine both e_2 and ν uniquely, subject to the original restriction.

COROLLARY. *Every point in the plane is an apse in some conic of the family.* For then $\psi_0 = \frac{\pi}{2}$, and equations (35) and (32) become $e_2 = \pm(Au_0 - \sqrt{Bu_0^2+1})$, and $\cos(\theta_0 - \nu) = \pm 1$, where the upper or lower sign is to be taken according as $A^2 \gtrless B + \rho_0^2$, — or $\rho_0^2 \lessgtr D$. Further, if $\rho_0^2 < D$, $\theta_0 = \nu$; while, if $\rho_0^2 > D$, $\theta_0 = \nu \pm \pi$. That is, any given point within the circle $\rho^2 = D$ is the pericenter in some conic of the family; and any point without that circle is the apocenter in some conic of the family. [This corrollary establishes the converse of a property obtained above.]

§ 12. *The Initial Conditions.*

It has been shown that, if the law (31) admits as trajectories any ellipses other than the original orbit, those ellipses must be members of the family of conics just discussed; and it is further necessary that the constants of areas have values determined as follows:

THE THIRD INVARIANT, E, or $h^2a^2(1-e^2)^2$. *The constant of areas must be proportional to the curvature at an apse.* For the radius of curvature at an apse is $a(1-e^2)$, so that the constant of areas h divided by the curvature is invariant.

Several questions now require consideration.

I. *Is an arbitrarily chosen constant of areas h_2 admissible for some conic of the family?* If so, then between h_2 and the required eccentricity e_2 must exist the relation $E = h_2^2\left(\beta_2 + \frac{\alpha_1^2 - \beta_2^2}{\gamma_2}\right) = h_2^2(Be_2^2 + D)$, which requires

$$h_2^2 B e_2^2 = E - h_2^2 D. \tag{36}$$

Equation (36) defines a unique positive number e_2, provided simply that $h_2^2 < \frac{E}{D}$, the value of h^2 in the circle of the family. [If $h_2 = 0$, $e_2 = \infty$, the curve being a straight line, as is indeed evident from other considerations.]

This result does not show, however, that there is an upper limit to the initial velocity of projection. For $h = v\rho \sin\psi$ constantly; and therefore arbitrarily large values of v yield admissible values of h, if only the initial distance or the angle ψ be sufficiently small.

II. *For an arbitrarily chosen constant of areas h_2*, less than the limit mentioned above, *what further restriction must be put upon the initial conditions in order that the trajectory be a conic?* Let the values of ρ, u, θ, v and ψ at the initial point be designated by the subscript zero. Now in order that the trajectory be a conic, e must have the value e_2, defined by (36).

Hence, in the first place, ρ_0 must lie between $a_2 - d_2$ and $a_2 + d_2$ [or ∞ if $e_2 \overset{=}{>} 1$]. Further ψ_0 must equal the value of ψ at the distance ρ_0 in the conic for which $e = e_2$. And, finally, v_0 must be given by $v_0 = h_2 u_0 \csc\psi_0$.

To formulate the second of these conditions, consider equation (25). Since, in any curve, $\tan\psi = -u\frac{d\theta}{du}$, or $u^2\csc^2\psi = u^2 + \left(\frac{du}{d\theta}\right)^2$, the value of $u_0^2\csc^2\psi_0$ in the orbit (25) is, for arbitrary conditions of projection,

$$u_0^2 \csc^2\psi_0 = \frac{k_2}{h_2^2} + \frac{E u_0 (2A\sqrt{Bu_0^2 + 1} - Bu_0)}{A^2 h_2^2 (A\sqrt{Bu_0^2 + 1} - Bu_0)^2}. \tag{37}$$

And in the conic for $e = e_2$, the value at $u = u_0$ is

$$u_0^2 \csc^2\psi = u_0^2 + \frac{(Bu_0^2 + 1)\left[e_2^2 - (Au_0 - \sqrt{Bu_0^2 + 1})^2\right]}{(A\sqrt{Bu_0^2 + 1} - Bu_0)^2}. \tag{38}$$

Equating the right members of (37) and (38), and observing that

$$E = h_2^2\left[B(e_2^2 - 1) + A^2\right],$$

a condition on $\frac{k_2}{h_2^2}$ is obtained. The irrational parts must be identical, likewise the rational parts:

$$\begin{aligned}\frac{k_2}{h_2^2}(-2A^3Bu_0)\sqrt{Bu_0^2+1} &= -2ABu_0(e_2^2-1)\sqrt{Bu_0^2+1}\\ \frac{k_2}{h_2^2}A^2[A^2+u_0^2(A^2B+B^2)] &= (e_2^2-1)[A^2+u_0^2(A^2B+B^2)].\end{aligned} \tag{39}$$

Each equation requires that $\frac{k_2}{h_2^2}=\frac{e_2^2-1}{A^2}$; that is, for all orbits of the family, $\frac{h^2(e^2-1)}{k}$ is *a fourth invariant*, restricting the initial conditions.

For a conic orbit with the constant of areas h_2, the velocity of projection at the distance ρ_0 must be v_0, where

$$v_0^2 = k_2 + F(u_0) = h_2^2\frac{e_2^2-1}{A^2} + F(u_0); \tag{40}$$

and ψ_0 is given by $h_2 = v_0\rho_0\sin\psi_0$. Or, reversing the order, ψ_0 may be obtained from (38), and then v_0 from either of the two equations last written.

III. *To determine the velocity, with which a particle projected at any point in any direction will describe a conic.* There is one, and only one, conic through the given point in the given direction. Since u_0 and ψ_0 are known, (35) determines e_2, real and positive; and (36) then determines h_2. Finally v_0 is given by (40) or by $v_0 = h_2 u_0 \csc\psi_0$. It is evident that there is one, and only one, real positive value of v_0 for which the orbit is a conic. The fact that there are initial velocities for which the orbits are other curves than conics, is important in a later section.

§ 13. *The Apsidal Angle in the Non-Conic Orbits.*

The question as to whether "the line of apsides advances or recedes," — that is, as to whether the apsidal angle Θ is greater than or less than π, — is a matter of interest. The expression for the force in laws of type I may be written, $f = u^2P(u)$, where

$$P(u) \equiv \frac{E}{(AR-Bu)^3}. \tag{41}$$

Now it has been shown elsewhere* that, if $P'(u) > 0$ for $u_1 > u > u_2$, then

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* F. L. Griffin: On the Apsidal Angle in Central Orbits, *Bulletin of the American Mathematical Society*, Vol. XIV (1907), pp. 6–16.

$\Theta > \pi$ in all orbits lying wholly between the circles $u = u_1$ and $u = u_2$; and conversely. Similarly, if $P'(u) < 0$, $\Theta < \pi$. And further, if $P'(u) < 0$, and decreases with u or is constant, then in any family of orbits, one of whose apsidal distances is constant, Θ increases with the second apsidal distance. Analogous statements apply in the opposite case.

For type I,

$$P'(u) = \frac{-3EB}{R(AR - Bu)^4}(Au - R). \tag{42}$$

Hence $P'(u) > 0$ if $R > Au$; that is, if $\rho^2 > D$; while $P'(u) < 0$ if $\rho^2 < D$. Consequently in all the orbits which lie wholly within the circle $\rho^2 = D$, $\Theta < \pi$; and in all orbits wholly without the circle $\rho^2 = D$, $\Theta > \pi$. [It is interesting to compare this result with the fact that in the conic orbits $\Theta = \pi$, and to note that a new proof is thus obtained for the earlier statement that every conic orbit admitted by the law (31) has its pericenter within and its apocenter without the circle $\rho^2 = D$.]

Moreover, since $P'\left(\frac{1}{\sqrt{D}}\right) = 0$ and $P'(u) < 0$ when $u < \frac{1}{\sqrt{D}}$, there exists a region within the circle $\rho^2 = D$ and without another concentric circle, throughout which Θ decreases as the apsidal distances in the orbits decrease The corresponding question for orbits outside $\rho^2 = D$ requires more extended computation.

§ 14. *A New Proof with Classic Hypotheses.*

From the foregoing discussion, it is possible to obtain a new proof of the operation of the Newtonian law in the binary systems, based upon the hypotheses mentioned in § 1 in connection with the classic theorem of Halphen and Darboux.

The classic hypotheses imply those of the present paper; for (I) and (II) are common, and, if every orbit is a conic, then the law according to which the force varies (not merely in one orbit but throughout the plane) is such as to permit real motion everywhere. And, since the operation of Newton's law follows from the new hypotheses, it follows also from the old ones.

Again, a new proof is possible with the assumptions of Halphen and Darboux modified as suggested in § 1. For the second theorem of Bertrand referred to may be applied, with assumption (III) and the fact that at least one conic orbit is an ellipse, to show that the force is central. Consider that ellipse; the center of force may lie only upon the major axis. For, if it were elsewhere, the use of (I) would furnish a law, for which some initial conditions would give rise to imaginary orbits, [§ 6], in violation of (III). Moreover, if the center of force were any point of the major axis save the center [excluded by assumption] or a focus, the corresponding law would admit imaginary orbits (if of type II) or real non-conic orbits (if of type I), [§ 12], again in violation of (III). Thus, Newton's law must operate.

§ 15. *Possible Modifications in Hypotheses.*

(1) As observed above, laws of type II might be excluded by strengthening hypothesis (III) to require the possibility of real motion, not merely sufficiently far from the center of force, but throughout the plane, an increase in severity which may seem on first consideration to be extremely small.

It is true, however, that the hypothesis in this revised form would exclude so important a law as that according to which an oblate spheroid attracts an exterior particle in its equatorial plane; for, if the latter law be required to operate within the spheroid, the force is imaginary within a certain interior circle.* But it is well to observe that even this exclusion would not be excessive in the present problem, inasmuch as the latter law admits as orbits no conics except circles.

Nevertheless, if the attracting body be regarded as having size, the severity of the revised hypothesis is greatly enhanced, as it would exclude the operation at exterior points of any law which, if operating also at interior points, would prescribe imaginary values of the force within the attracting body, even though all (physically possible) real initial conditions give rise to real orbits. If modified to require only the reality of the force everywhere outside the attracting body, the hypothesis would not suffice to exclude laws of type II for which $ae > d > ae^2$, and would suffice to exclude those where $d < ae^2$ only when strengthened by the assumption that the bedies do not collide. [These latter statements rest upon the fact that, if ρ_1 denote the value of ρ for which the imaginary radical vanishes, $\rho_1 < a - d$ (the least radius vector in the orbit) if $ae > d > ae^2$, and $\rho_1 =$ the minimum normal if $d < ae^2$.] When $ae > d > ae^2$, however, the center of force would at least have to be sufficiently near the focus to have the principal star contain the focus; for the inequality $\rho_1 > ae - d$ follows from the known inequality $0 > (d - ae)\,(d + ae - 2ae^3)$.

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* For the law in question see Moulton. *loc. cit.*, p. 123, or paper cited in *Bulletin*, p. 13.

(2) Again, it has been noted that all exterior points, and all points on the ellipse except the ends of the major axis, could be excluded without the aid of the reference cited in § 1. These two, and all other, points of the curve might be excluded by formulating as an assumption for material points the physical principle of impenetrability, or by otherwise prohibiting collision. This change would, however, involve an altogether unnecessary (even if mild) addition to the assumptions of the paper.

(3) Finally, in any non-rigid body tides would be formed with a resulting difference in the attracting force, which, considered as a perturbation, would in general forbid the constancy of $\frac{h^2(e^2-1)}{k}$ (the fourth invariant). Doubtless a substitute for hypothesis (II) can be framed, based upon some such requirement as that, under tidal disturbance the force shall cause the velocity at any instant to be such as to permit the description of a conic as the subsequent orbit were tidal disturbance to disappear. In addition, it would be necessary to require that the center of force be not at the center of the orbit, in order to exclude the law of the direct distance which compels an elliptic orbit for all initial conditions.

WILLIAMS COLLEGE, *September*, 1908.